Logarithmic Voronoi polytopes for discrete linear models

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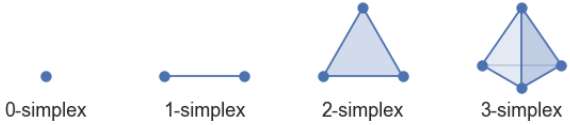
Basic definitions

- A *probability simplex* is defined as

$$\Delta_{n-1} = \{(p_1, \ldots, p_n) \, : \, p_1 + \cdots + p_n = 1, p_i \geq 0 \text{ for } i \in [n]\}.$$



- An *algebraic statistical model* is a subset $\mathcal{M} = \mathcal{V} \cap \Delta_{n-1}$ for some variety $\mathcal{V} \subseteq \mathbb{C}^n$.
- For an empirical data point $u = (u_1, ..., u_n) \in \Delta_{n-1}$, the *log-likelihood function* defined by u assuming distribution $p = (p_1, ..., p_n) \in \mathcal{M}$ is

$$\ell_u(p) = u_1 \log p_1 + u_2 \log p_2 + \cdots + u_n \log p_n + \log(c).$$

Maximum likelihood estimation

1. The maximum likelihood estimation problem (MLE):

Given a sampled empirical distribution $u \in \Delta_{n-1}$, which point $p \in \mathcal{M}$ did it most likely come from? In other words, we wish to maximize $\ell_u(p)$ over all points $p \in \mathcal{M}$.

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2. Computing logarithmic Voronoi cells:

Given a point $q \in \mathcal{M}$, what is the set of all points $u \in \Delta_{n-1}$ that have q as a global maximum when optimizing the function $\ell_u(p)$ over $\mathcal{M}$?

We call the set of all such elements $u \in \Delta_{n-1}$ above the *logarithmic Voronoi cell* at q.

Proposition (A., Heaton)

Logarithmic Voronoi cells are convex sets.

The *log-normal space* at q is the space of possible data points $u \in \mathbb{R}^n$ for which q is a critical point of $\ell_u(p)$. It is a *linear* space.

Intersecting this space with the simplex Δ_{n-1}, we obtain a polytope, which we call the *log-normal polytope* at q.

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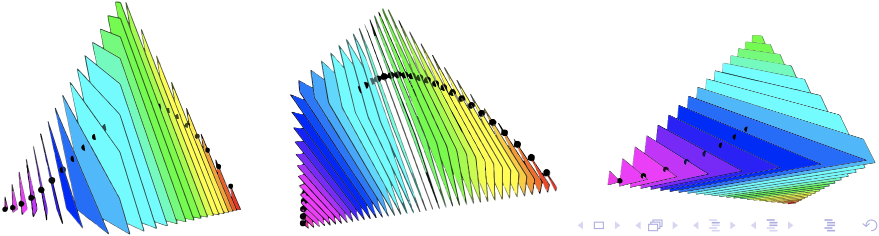
Example (The twisted cubic.)

The curve is given by $p \mapsto \left(p^3, 3p^2(1-p), 3p(1-p)^2, (1-p)^3\right)$.



Discrete linear models

A *linear model* is given parametrically by nonzero linear polynomials.

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Any d-dimensional linear model inside Δ_{n-1} can be written as

$$\mathcal{M} = \{c - Bx : x \in \Theta\}$$

where B is a $n \times d$ matrix, whose columns sum to 0, and $c \in \mathbb{R}^n$ is a vector, whose coordinates sum to 1.

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A *co-circuit* of B is a vector $v \in \mathbb{R}^n$ of minimal support such that $vB = 0$. A co-circuit is *positive* if all its coordinates are positive.

We call a point $p = (p_1, \ldots, p_n) \in \mathcal{M}$ is *interior* if $p_i > 0$ for all $i \in [n]$.

Interior points

For an interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell at p is the set

$$\log \mathrm{Vor}_{\mathcal{M}}(p) = \left\{ r \cdot \mathrm{diag}(p) \in \mathbb{R}^n : rB = 0,\ r \geq 0,\ \sum_{i=1}^{n} r_i p_i = 1 \right\}.$$

Proposition (A.)

For any interior point $p \in \mathcal{M}$, the vertices of $\log \mathrm{Vor}_{\mathcal{M}}(p)$ are of the form $v \cdot \mathrm{diag}(p)$ where v are unique representatives of the positive co-circuits of B such that $\sum_{i=1}^{n} v_i p_i = 1$.

Gale diagrams

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a vector configuration in $\mathbb{R}^d$, whose affine hull has dimension d. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}.$$

Let $\{B_1, \ldots, B_{n-d-1}\}$ be a basis for $\ker(A)$ and $B := [B_1 \; B_2 \; \cdots \; B_{n-d-1}]$. The configuration $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ of row vectors of B is the *Gale diagram* of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

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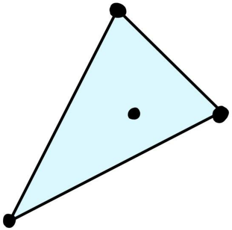
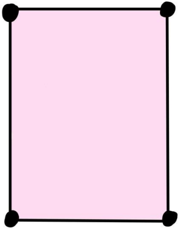
Theorem (A.)

For any interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell at p is combinatorially isomorphic to the dual of the polytope obtained by taking the convex hull of a vector configuration with Gale diagram B.

Corollary

Logarithmic Voronoi cells of all interior points in a linear model have the same combinatorial type.

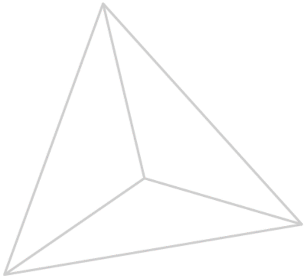




Proposition (A.)

Every $(n-d-1)$*-dimensional polytope with at most* n *facets appears as a logarithmic Voronoi cell of a* d*-dimensional linear model inside* Δ_{n-1}.

Examples



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$c = (1/4, 1/4, 1/4, 1/4)$

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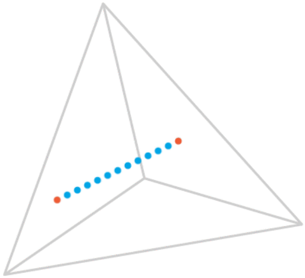

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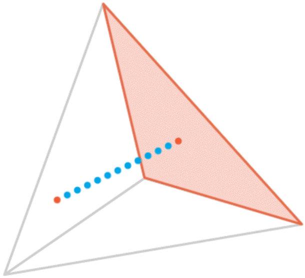

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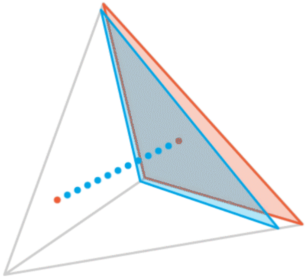

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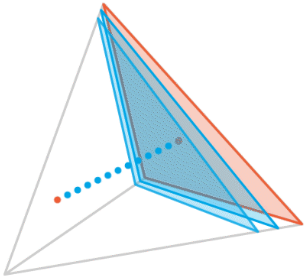

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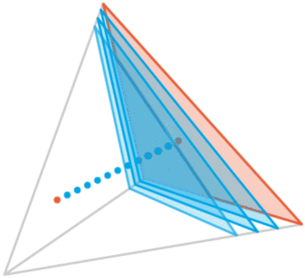

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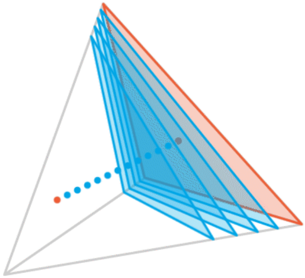

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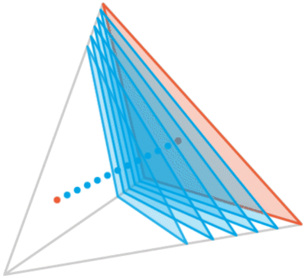

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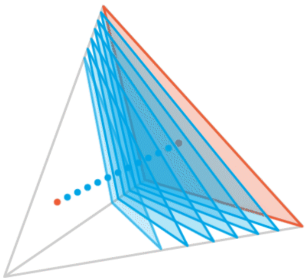

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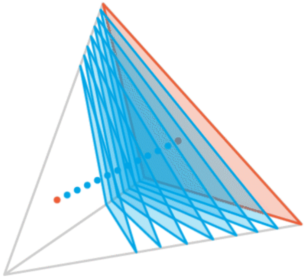

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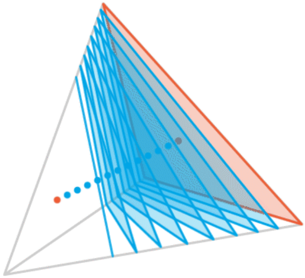

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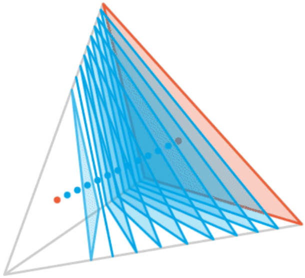

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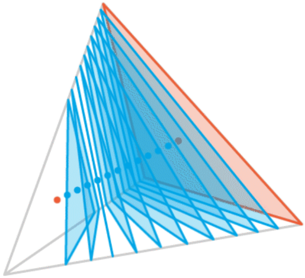

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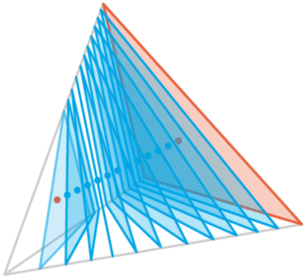

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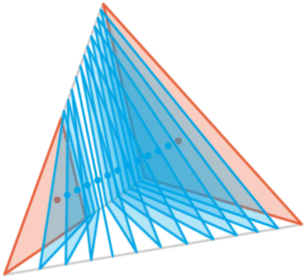

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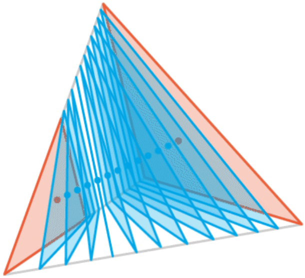

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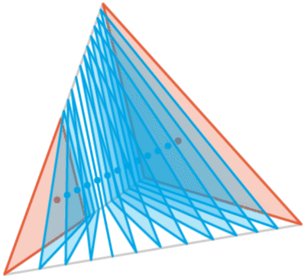

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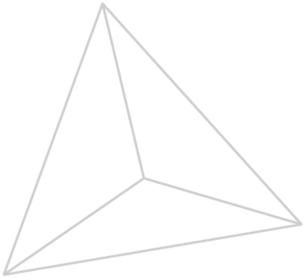

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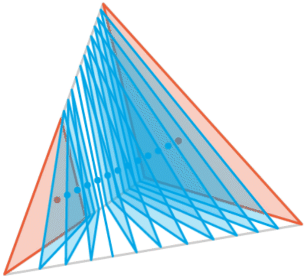

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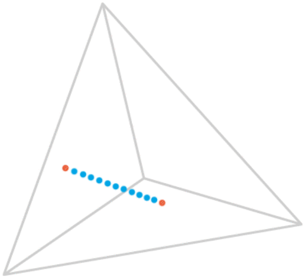

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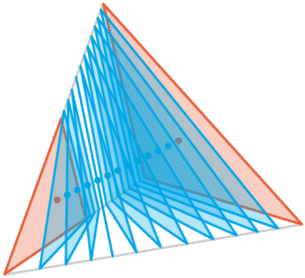

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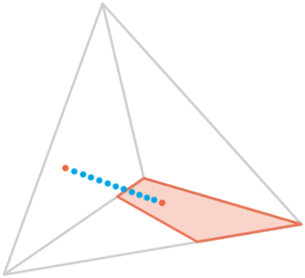

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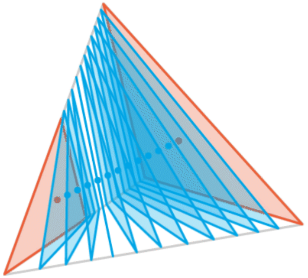

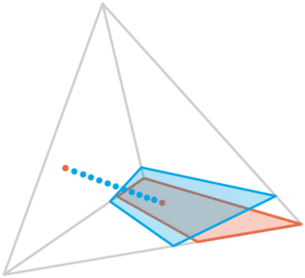

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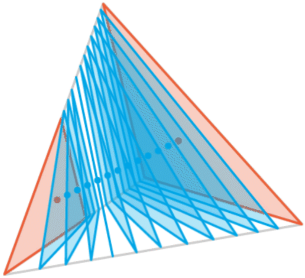

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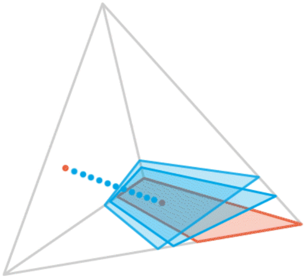

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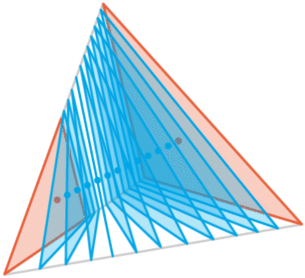

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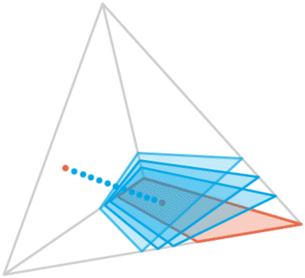

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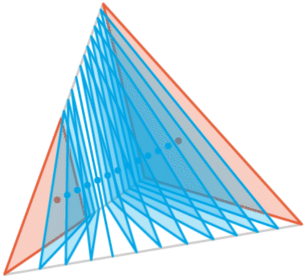

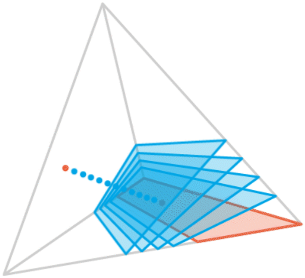

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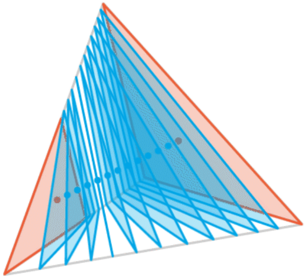

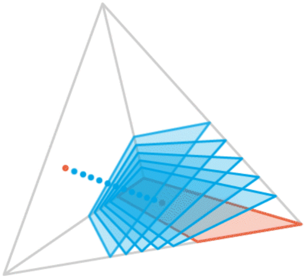

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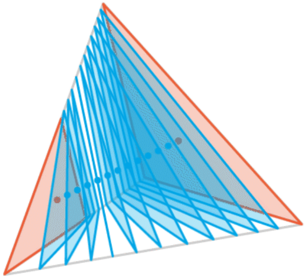

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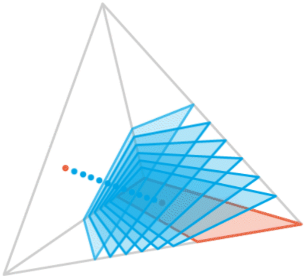

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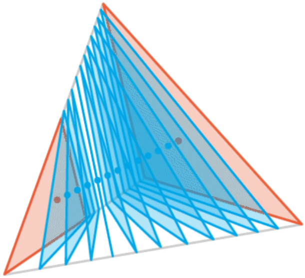

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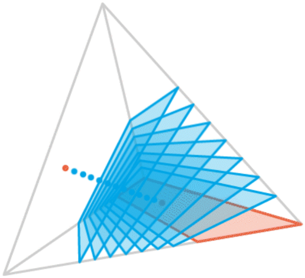

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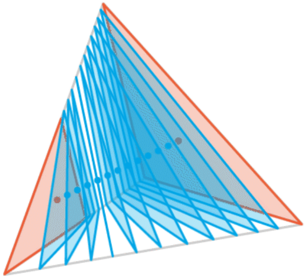

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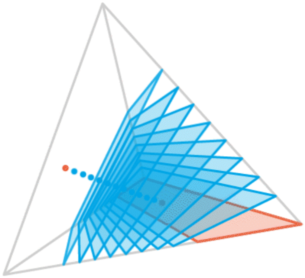

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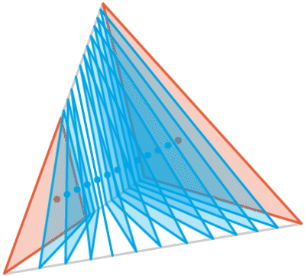

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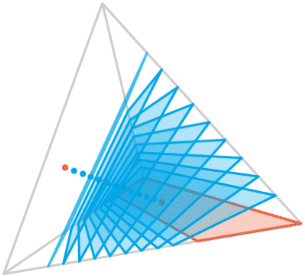
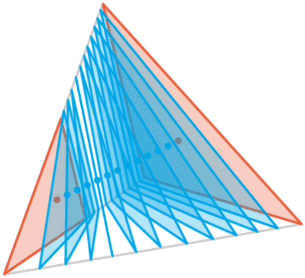
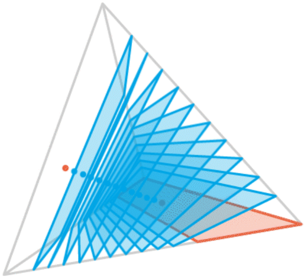


$B = [-2, -1, 1, 2]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

Examples

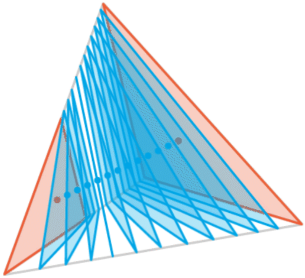


$B = [1, -5, 3, 1]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

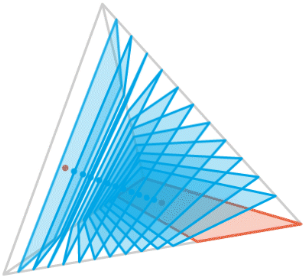


$B = [-2, -1, 1, 2]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

Examples

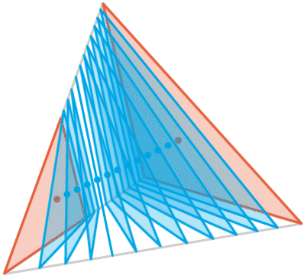


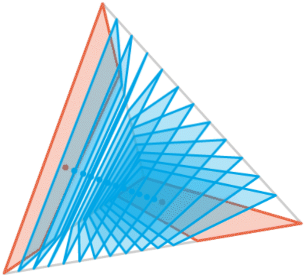
$B = [1, -5, 3, 1]^T$
$c = (1/4, 1/4, 1/4, 1/4)$



$B = [-2, -1, 1, 2]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

Examples





$B = [1, -5, 3, 1]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

$B = [-2, -1, 1, 2]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

On the boundary

Theorem (A.)

Let $\mathcal{M}$ be the d-dimensional linear model, obtained by intersecting the affine linear space L with Δ_{n-1}. Let $w \in \mathcal{M}$ be a point on the boundary of the simplex. If L intersects Δ_{n-1} transversally, then the logarithmic Voronoi polytope at w has the same combinatorial type as those at the interior points of $\mathcal{M}$.

Example: $d = 1$.

Let $\mathcal{M}$ be a 1-dimensional linear model inside the simplex Δ_{n-1}. Then $\mathcal{M} = \{c - Bx : x \in \Theta\}$, where

$$B = [\underbrace{b_1 \ \dots \ b_m}_{>0} \ \underbrace{b_{m+1} \ \dots \ b_n}_{<0}]^T \text{ and } c = (c_i).$$

Then Θ is the interval $[x_\ell, x_r] = [c_\ell/b_\ell, c_r/b_r]$ where $b_\ell < 0$ and $b_r > 0$. Assume $r = 1$. The log-Voronoi cell at x_r is the polytope at the boundary of Δ_{n-1} with the vertices

$$\{e_j : b_j < 0\} \cup \left\{ \underbrace{\frac{(c_i - b_i(c_1/b_1))b_j}{b_j c_i - b_i c_j} e_i - \frac{(c_j - b_j(c_1/b_1))b_i}{b_j c_i - b_i c_j} e_j}_{v_{ij}} : \substack{i \neq r, \\ b_i > 0, \\ b_j < 0} \right\}.$$

The log-Voronoi cell at x_ℓ is described similarly.

Thanks!